LINEAR DIFFERENTIAL EQUATIONS OVER ARBITRARY ALGEBRAICALLY CLOSED FIELDS

SEVER ANGEL POPESCU

ABSTRACT. Let K be an arbitrary algebraically closed field of characteristic zero and let $K[[x]]$ be the ring of integral formal power series; let Ω be the K-subalgebra of $K[[x]]$ generated by x and the subset $T_K = \{\exp(\lambda x) : \lambda \in K\}$. In this note we supply some easy and elegant proofs for some classical results on the preimage of elements of the form $x^q Q(x)\exp(rx)$ through a linear differential operator with coefficients in K. We also make some theoretical considerations on the structure of the space of all solutions for a linear ODE defined over $K[[x]]$.

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1. SOME INTRODUCTORY REMARKS

Let K be an algebraically closed field of characteristic zero [LS] and let x be a variable over K (simply an element x not belonging to K). Let $K[[x]]$ be the ring of all formal integral power series $f = a_0 + a_1 x + \ldots$, $a_i \in K$, $i = 0, 1, \ldots$. Here $\frac{d}{dx} : K[[x]] \to K[[x]]$, $\frac{df}{dx} = a_1 + 2a_2 x + \ldots + n a_n x^{n-1} + \ldots$, is the usual differential operator defined on $K[[x]]$. For $y \in K[[x]]$, we also denote

$$y^{(n)} = \frac{d^n y}{dx^n} = \left(\underbrace{\frac{d}{dx} \circ \ldots \circ \frac{d}{dx}}_{n-times} \right)(y).$$

Theorem 1. *Let $b_1, \ldots, b_n$, c be $n+1$ elements in $K[[x]]$. Then any solution $y \in K[[x]]$ of the linear differential equation*

$$y^{(n)} + b_1 y^{(n-1)} + \ldots + b_n y = c, \tag{1.1}$$

is of the form:

$$y = d_0 + d_1 x + \ldots + d_t x^t + \ldots, \tag{1.2}$$

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where $d_0, ..., d_{n-1}$ *are* n *free parameters in* K *and*

$$d_{n+k} = \sum_{j=0}^{k} \alpha_j^{(k)} c_j + \sum_{i=0}^{n-1} \beta_i^{(k)} d_i, \tag{1.3}$$

$k = 0, 1, ..., \alpha_j^{(k)}, \beta_i^{(k)} \in \mathbb{Q}[b_{uv}],$ *where*

$$b_u = b_{u0} + b_{u1}x + ..., \tag{1.4}$$

$$c = c_0 + c_1 x + ..., \tag{1.5}$$

$b_{uv}, c_j \in K,\ u = 1, 2, ..., n,\ v = 0, 1, ...$.

Proof. For y of the form (1.2) one has:

$$y^{(m)} = \sum_{t=m}^{\infty} t(t-1)...(t-m+1) d_t x^{t-m}, \tag{1.6}$$

$m = 1, 2, ..., m$. Let us prove (1.3) by mathematical induction. In order to prove (1.3) for $k = 0$, we consider the equality (1.1) modulo x, i.e. we look at it in the quotient ring $K[[x]]/(x)$, where (x) is the ideal generated by x in $K[[x]]$. Thus we get:

$$n! d_n + b_{10}(n-1)! d_{n-1} + ... + b_{n0} d_0 = c_0. \tag{1.7}$$

Since the characteristic of K is equal to 0 ($\mathbb{Q} \subset K$), we obtain:

$$d_n = \alpha_0^{(0)} c_0 + \sum_{i=0}^{n-1} \beta_i^{(0)} d_i,$$

where $\alpha_0^{(0)} = \frac{1}{n!}$, $\beta_i^{(0)} = -\frac{i!}{n!} b_{n-i,0} d_i$, $i = 0, 1, ..., n-1$. We assume now that formula (1.3) is true for $k = 0, 1, ..., q$. Let us prove it for $k = q+1$. For this, we come back to (1.1) and consider it modulo x^{q+2} :

$$\sum_{t=n}^{n+q+1} \frac{t!}{(t-n)!} d_t x^{t-n} + \left[\sum_{l=0}^{q+1} b_{1l} x^l\right] \left[\sum_{t=n-1}^{n+q} \frac{t!}{(t-n+1)!} d_t x^{t-n+1}\right] +$$

$$+ ... + \left[\sum_{l=0}^{q+1} b_{nl} x^l\right] \left[\sum_{t=0}^{q+1} d_t x^t\right] = c_0 + c_1 x + ... + c_{q+1} x^{q+1} \tag{1.8}$$

In this last polynomial equality we identify the coefficients of x^{q+1} and get:

$$\frac{(n+q+1)!}{(q+1)!} d_{n+q+1} + \sum_{l=0}^{q+1} \frac{(n+q-l)!}{(q-l+1)!} b_{1l} d_{n+q-l} + ...+$$

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$$+\sum_{l=0}^{q+1} b_{nl} d_{q+1-l} = c_{q+1}. \tag{1.9}$$

Since in these above sums appear (excepting the constants!) only $d_0, ..., d_n, ..., d_{n+q}$ and since $d_n, d_{n+1}, ..., d_{n+q}$ are computed by using formulas (1.3) - mathematical induction hypotheses - we finally conclude that d_{n+q+1} can be written as in (1.3). If $c = 0$, i.e. $c_0 = c_1 = ... = 0$, we see from (1.3) that d_{n+k} are linear combinations of $d_0, ..., d_{n-1}$, i.e. the vector space of all solutions of the homogenous equation (1.1) (with $c = 0$) is exactly equal to n. □

Remark 1. If, in (1.1), $c = 0$, i.e. $c_0 = c_1 = ... = 0$, we see from (1.3) that d_{n+k} are linear combinations of $d_0, ..., d_{n-1}$, i.e. the vector space of all solutions of the homogenous equation (1.1) (with $c = 0$) is exactly equal to n.

Remark 2. Theorem 1 can be viewed as an existence and uniqueness theorem for the following Cauchy problem in $K[[x]]$: find $y = y(x) \in K[[x]]$ such that the equality (1.1) is true and $y(0) = d_0$, $y'(0) = d_1, ..., y^{(n-1)}(0) = (n-1)!d_{n-1}$ are given. This means to fix $d_0, ..., d_{n-1}$ in (1.2). Since $d_n, d_{n+1}, ...$ are linear functions of $d_0, ..., d_{n-1}$, $c_0, c_1, ...$ (see (1.3)), the existence and uniqueness of y is clear enough. Here, for $f \in K[[x]]$, $f(0)$ means the constant term of f

For any element $\lambda \in K$ we denote $\exp(\lambda x)$ the following formal power series:

$$\exp(\lambda x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} x^n \tag{1.10}$$

with coefficients in K. Since $\mathbb{Q} \subset K$, one can easily see that

$$\exp(\lambda x) \cdot \exp(\mu x) = \exp((\lambda + \mu)x),$$

for any $\lambda, \mu \in K$.

Let us denote $T_K = \{\exp(\lambda x) : \lambda \in K\}$ and let Ω be the K-subalgebra of $K[[x]]$ generated by the variable x and by the subset T_K. In the following we shall consider linear differential equations of the form (1.1), with $c = 0$ and $b_1, ..., b_n$ constant elements, i.e. elements in K.

Let I be the identity operator defined on Ω and let $L : \Omega \to \Omega$ be the following linear differential operator:

$$L[y] = y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1} y' + a_n y, \tag{1.11}$$

where $y \in \Omega$, $y^{(j)} = \frac{d^j y}{dx^j}$ and $a_j \in K$ for any $j = 1, 2, ..., n$. Here $\frac{d^0}{dx^0} = I$.

The equation

$$E(z) = z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n = 0 \tag{1.12}$$

is called the associated algebraic equation to the differential operator L. Let $r_1, r_2, ..., r_n \in K$ be all the roots of equation (1.12). After an eventually appropriate rearrangement of $r_1, r_2, ..., r_n$, let $r_1, r_2, ..., r_k$, $k \leq n$, be all the distinct roots with their algebraic multiplicities $t_1, t_2, ..., t_k$ respectively. So

$$E(z) = (z - r_1)^{t_1} (z - r_2)^{t_2} ... (z - r_k)^{t_k} \tag{1.13}$$

is the decomposition of the polynomial $E(z)$ in coprime factors over the algebraically closed field K. The decomposition (1.13) induces the following decomposition of the linear operator L into powers of linear differential operators of order one:

$$L = \left(\frac{d}{dx} - r_1 I \right)^{t_1} \left(\frac{d}{dx} - r_2 I \right)^{t_2} ... \left(\frac{d}{dx} - r_k I \right)^{t_k} \tag{1.14}$$

In this note we supply new easy elementary proofs (they can be used even in the undergraduate teaching!) for the following classical results (known for $K = \mathbb{C}$, the field of complex numbers).

Theorem 2. *Let $\lambda \in K \backslash \{r_1, r_2, ..., r_k\}$ and let $Q \in K[x]$, $degQ = m$ be an arbitrary polynomial of degree m with coefficients in the field K. Then there exists a unique polynomial $P \in K[x]$ of degree m such that*

$$L[P \exp(\lambda x)] = Q \exp(\lambda x). \tag{1.15}$$

Theorem 3. *Let q be a natural number (0 included) and let $Q \in K[x]$ be a polynomial of degree m. Then there exists a unique polynomial $P_q \in K[x]$ of degree m such that*

$$L\left[x^{q+t_1} P_q \exp(r_1 x)\right] = x^q Q \exp(r_1 x), \tag{1.16}$$

where r_1 is a root of algebraic multiplicity t_1 of the above equation (1.12). In particular, if $q = 0$, we get a classical result in LDE of order n with constant coefficients.

2. Proof of Theorems 2, 3 and some other remarks

Lemma 1. *Let $\lambda, r \in K$, $\lambda \neq r$ and let $Q \in K[x]$ be a polynomial of degree m with coefficients in K. Then there exists a unique polynomial $P \in K[x]$ of degree m such that*

$$\left(\frac{d}{dx} - rI \right) [P \exp(\lambda x)] = Q \exp(\lambda x). \tag{2.1}$$

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If K_0 is a subfield of K, if $\lambda, r \in K_0$ and if $Q \in K_0[x]$, then this polynomial P has coefficients in K_0.

Proof. Let us compute the left side of (2.1) and then simplify by $\exp(\lambda x)$. We get:

$$P' + (\lambda - r)P = Q. \tag{2.2}$$

If we write $Q(x) = c_0 + c_1 x + ... + c_m x^m$, from (2.2) one can easily uniquely determine the coefficients $b_0, b_1, ..., b_m$ of the searched for polynomial $P(x) = b_0 + b_1 x + ... + b_m x^m$. Moreover, we see that each b_j is a linear expression of $c_0, c_1, ..., c_m$ with coefficients in $\mathbb{Q}(\lambda, r)$ which is included in any subfield K_0 of K with $\lambda, r \in K_0$. Here $\mathbb{Q}$ is the subfield of rational numbers viewed in K (since the characteristic of K is equal to zero). □

Lemma 2. *Let $r \in K$, $Q \in K[x]$, $\deg Q = m$ and $q \in \mathbb{N}$. Then there exists a unique polynomial $P_q \in K[x]$, $\deg P_q = m$, such that:*

$$\left(\frac{d}{dx} - rI\right)\left[x^{q+1} P_q \exp(rx)\right] = x^q Q \exp(rx). \tag{2.3}$$

If $r \in K_0$, a subfield of K, and $Q \in K_0[x]$, the $P_q \in K_0[x]$.

Proof. Computing the left side of (2.3) and then simplifying by $x^q \exp(rx)$, one obtains:

$$(q+1)P_q + xP_q' = Q. \tag{2.4}$$

If we write again $Q(x) = c_0 + c_1 x + ... + c_m x^m$, it is easy to uniquely determine the coefficients $d_0, d_1, ..., d_m$ of the searched polynomial $P_q(x) = d_0 + d_1 x + ... + d_m x^m$ as linear combinations of $c_0, c_1, ..., c_m$ with coefficients in $\mathbb{Q}(r)$ which is included in any subfield of K which contains r. □

Now we can prove Theorem 2.

Proof. (Theorem 2) Mathematical Induction is used on $n = t_1 + t_2 + ... + t_k$ (see the above notation). For $n = k = t_1 = 1$, one directly apply Lemma 1. Assume that the statement of Theorem 2 is true for $h = 1, 2, ..., n-1$. Let us prove it for $h = n$. Let

$$L = \left(\frac{d}{dx} - r_1 I\right) L^*,$$

where

$$L^* = \left(\frac{d}{dx} - r_1 I\right)^{t_1 - 1} \left(\frac{d}{dx} - r_2 I\right)^{t_2} ... \left(\frac{d}{dx} - r_k I\right)^{t_k}. \tag{2.5}$$

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Let $P_1 \in K[x]$ be the unique polynomial of degree m with

$$\left(\frac{d}{dx} - r_1 I\right)[P_1 \exp(\lambda x)] = Q \exp(\lambda x). \tag{2.6}$$

(see Lemma 1). Let now $P \in K[x]$ be the unique polynomial of degree m such that

$$L^* [P \exp(\lambda x)] = P_1 \exp(\lambda x). \tag{2.7}$$

(see the above Mathematical Induction assumption).

Apply now $\left(\frac{d}{dx} - r_1 I\right)$ in both sides in (2.7) and finally obtain

$$L[P \exp(\lambda x)] = Q \exp(\lambda x),$$

i.e. the statement of Theorem 2. □

Let now prove Theorem 3.

Proof. (Theorem 3) The same Mathematical Induction on $n = t_1 + t_2 + ... + t_k$ is used. For $n = k = t_1 = 1$, one can simply apply Lemma 2. Suppose that the statement of Theorem 3 is true for $h = 1, 2, ..., n-1$. Let us prove it for $h = n$. Let again

$$L = \left(\frac{d}{dx} - r_1 I\right) L^*,$$

where L^* is defined in (2.5). Let $P_{1,q} \in K[x]$ be the unique polynomial of degree m with

$$\left(\frac{d}{dx} - r_1 I\right)\left[x^{q+1} P_{1,q} \exp(r_1 x)\right] = x^q Q \exp(r_1 x).$$

(see Lemma 2).

Let now $P \in K[x]$ be the unique polynomial of degree m with

$$L^* \left[x^{q+1+t_1-1} P_q \exp(r_1 x)\right] = x^{q+1} P_{1,q} \exp(r_1 x). \tag{2.8}$$

Let us apply $\left(\frac{d}{dx} - r_1 I\right)$ both sides in (2.8). Finally we get that P_q is the unique required polynomial in (1.16) and the proof of Theorem 3 is complete. □

Let us say some words on the "structure of solutions" of the equation $L[y] = 0$ in this general frame.

Lemma 3. *Let r_1 be a root of algebraic multiplicity t_1 of the polynomial equation $E(z) = 0$ (see 1.12). Then for any polynomial $P(x) \in K[x]$ with $\deg P < t_1$ one has that $L[P \exp(r_1 x)] = 0$.*

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Proof. We simply successively compute

$$\left(\frac{d}{dx} - r_1 I\right)[P(x)\exp(r_1 x)] = P'(x)\exp(r_1 x),$$

$$\left(\frac{d}{dx} - r_1 I\right)^2 [P(x)\exp(r_1 x)] = \left(\frac{d}{dx} - r_1 I\right)[P'(x)\exp(r_1 x)] =$$
$$= P''(x)\exp(r_1 x), ...$$
$$... \left(\frac{d}{dx} - r_1 I\right)^{t_1} [P(x)\exp(r_1 x)] = P^{(t_1)}(x)\exp(r_1 x) = 0,$$

because the t_1-th derivative of a polynomial of degree less than t_1 is equal to zero. □

Lemma 4. *Let $r_1, r_2, ..., r_k$ be k distinct elements in K. Then* $\exp(r_1 x)$, $\exp(r_2 x), ..., \exp(r_k x)$ *are (as elements in the vector space Ω over $K(x)$) linear independent over $K(x)$. Here $K(x)$ is the rational function field in the variable x over the initial field K.*

Proof. Let

$$P_1(x)\exp(r_1 x) + P_2(x)\exp(r_2 x) + ... + P_k(x)\exp(r_k x) = 0, \tag{2.9}$$

where we can assume that $P_1(x)$, $P_2(x)$, ..., $P_k(x)$ are nonzero polynomials of degrees $n_1, n_2, ..., n_k$ respectively. We want to prove that all these polynomials are zero. Let $n = n_1 + n_2 + ... + n_k$ be the sum of all degrees of these polynomials. Since we just assumed that all polynomials are nonzero, we have that $n \geq 0$. From the set of all linear combinations like in (2.9) we choose one with n the least possible. We shall prove firstly that $n = 0$. One can also assume that $k \geq 2$, otherwise, from (2.9), $P_1(x) = 0$ and so $n = -\infty$. Take for instance $k_1 \neq 0$. If $n > 0$, let us differentiate the equality (2.9) with respect to x :

$$\sum_{i=1}^{k} (P_i' + r_i P_i)\exp(r_i x) = 0. \tag{2.10}$$

Since $r_1 \neq 0$, we can eliminate $P_1 \exp(r_1 x)$ between (2.9) and (2.10) and obtain:

$$-\frac{1}{r_1} P_1' \exp(r_1 x) + \sum_{i=2}^{k} \left(P_i - \frac{1}{r_1} P_i' - \frac{r_i}{r_1} P_i\right)\exp(r_i x) = 0.$$

Here is an expression like that one from (2.9) but the sum of the degrees of the corresponding polynomials is at most $n-1$. Thus we just obtained a contradiction relative to the choice of n. Hence $n = 0$, i.e. any nontrivial linear combination between $\exp(r_1 x), \exp(r_2 x), ..., \exp(r_k x)$, over $K(x)$ must have coefficients in K. It remains to prove that from any linear combination

$$a_1 \exp(r_1 x) + a_2 \exp(r_2 x) + ... + a_k \exp(r_k x) = 0, \tag{2.11}$$

one derives that $a_1 = 0$, $a_2 = 0, ..., a_k = 0$. We prove this by mathematical induction on k. The statement is clear for $k = 1$. Suppose that the statement is true for $k - 1$. Let us prove it for k, i.e. we consider the equality (2.11). Let us differentiate (2.11):

$$r_1 a_1 \exp(r_1 x) + r_2 a_2 \exp(r_2 x) + ... + r_k a_k \exp(r_k x) = 0. \tag{2.12}$$

Let us now multiply (2.11) by r_1 and subtract the result from (2.12):

$$(r_2 - r_1) a_2 \exp(r_2 x) + ... + (r_k - r_1) a_k \exp(r_k x) = 0.$$

By using the induction hypothesis we get:

$$(r_2 - r_1) a_2 = 0, ..., (r_k - r_1) a_k = 0.$$

Since $r_1, r_2, ..., r_k$ are distinct, we conclude that $a_2 = 0$, $a_3 = 0$, ..., $a_k = 0$. Coming back to (2.11) we find that $a_1 = 0$ (we proved above that $\exp(\lambda x) \neq 0$) and the proof of the lemma is completed. □

Theorem 4. *Let S be the K-vector subspace of Ω which consists of all solutions $y \in \Omega$ of the linear differential equation $L[y] = 0$. Then* $\dim_K S = n$.

Proof. Since for any $i = 1, 2, ..., k$ and for any polynomial P_i of degree $t_i - 1$ one has $L[P_i \exp(r_i x)] = 0$ (see Lemma 3) any element

$$y = \sum_{i=1}^{k} P_i \exp(r_i x) \in \Omega$$

is a solution of $L[y] = 0$. Since also $\exp(r_1 x)$, $\exp(r_2 x), ...,$ $\exp(r_k x)$ are linear independent over $K(x)$, we conclude that $\{x^{j_i} \exp(r_i x)\}$, $i = 1, 2, ..., k$ and $j_i = 0, 1, ..., t_{i-1}$ are linear independent solutions of $L[y] = 0$, over K. Thus $\dim_K S \geq n = \sum_{i=1}^{k} t_i$. But, from Remark 1, $\dim_K S \leq n$ and thus $\dim_K S = n$. □

Example 1. Let $K = \overline{\mathbb{Q}}_3$ be a fixed algebraic closure (see [LS]) of the 3-adic number field $\mathbb{Q}_3$ (see [GP] or [G] for definitions and notation) and let

$$y'' + 2y = 0 \tag{2.13}$$

be a linear differential equation of order two. If we search for a solution $y \in K[[x]]$ like in Theorem 1, we can easily find two linear independent (over K) power series $y_1, y_2 \in \mathbb{Q}[[x]]$, solutions of (2.13), such that the K-vector space of all solutions in $K[[x]]$ of (2.13) can be generated

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by y_1 and y_2. Usually, it is not so comfortable to work with solutions in this form. The best idea is to search for solutions of the form: $y^* = \exp(\lambda x) \in \Omega$, $\lambda \in K$. Since $(y^*)' + 2y^* = 0$ implies $\lambda^2 + 2 = 0$, this means that $\lambda = \pm i\sqrt{2}$, where $i = \sqrt{-1}$, $\sqrt{2}$ are in K. They, separately, are not in $\mathbb{Q}_3$, but $\pm i\sqrt{2} \in \mathbb{Q}_3$ (see [GP] or [G]), because the algebraic equation $\lambda^2 + 2 = 0$ has two distinct solutions $(\pm\widehat{1})$ modulo 3. Thus, $y_1^* = \exp(i\sqrt{2})$ and $y_2^* == \exp(-i\sqrt{2})$ are linear independent solutions of (2.13) which generate the whole K-vector space of all solutions, i.e. any solution y is of the form:

$$y = C_1 y_1^* + C_2 y_2^*,$$

where C_1, C_2 are arbitrary elements in K. This last description is easier than that one given in language of power series.

Example 2. Let this time $K = \overline{\mathbb{C}(X)}$ be a fixed algebraic closure of the field $\mathbb{C}(X)$ of rational functions with coefficients in the field of complex numbers. Let us consider the linear differential equation

$$y'' + Xy = 0,$$

with coefficients in K. It is very easy to find two linear independent power series $y_1, y_2 \in \mathbb{Q}(X)[[x]]$ which generate the 2-dimensional vector space of all solutions of the differential equation, viewed in $K[[x]]$. However, by using the above theory, we can find two linear independent solutions $y_1^* = \exp(\sqrt{X} \cdot x)$, $y_2^* = \exp(-\sqrt{X} \cdot x)$ which can generate the same 2-dimensional vector space of all solutions, but in an easier way. Here $\sqrt{X}$, $-\sqrt{X} \in K$ because of the Newton-Puisseux theorem $(K = \cup_{n=1}^{\infty} \mathbb{C}((X^{\frac{1}{n}})))$, [Ab].

The above easy, elementary and elegant treatment of the linear differential equations with constant coefficients can be used for $K = \mathbb{C}$. An alternate treatment in this case one can find in [RR].

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Technical University of Civil Engineering Bucharest, Department of Mathematics and Computer Science, B-ul Lacul Tei 122, sector 2, Bucharest 020396, OP 38, Bucharest, Romania
E-mail address: angel.popescu@gmail.com

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