A NOTE ON EXTENSIONS OF OPEN SETS BY IDEALIZATIONS

AHU ACIKGOZ AND TAKASHI NOIRI

Abstract. Recently, Abbas [1] has introduced and investigated the notion of h-open sets in a topological space. As a generalization of h-open sets, in [2] we introduced hI-open sets in an ideal topological space (X, τ, I) and obtained some properties of hI-open sets. In this paper, we introduce and investigate $h^\star$-open sets on an ideal topological space. We show that $h^\star$-open sets lie between open sets and hI-open sets and $h^\star$-open sets are indepenent of h-open sets.

Mathematics Subject Classification (2010): 54A05, 54A10, 54C10
Key words: h-open, hI-open, $h^\star$-open, $h^\star$-continuous, $h^\star$-irresolute, ideal topological space.

Article history:
Received: June 10, 2023
Received in revised form: July 29, 2023
Accepted: July 31, 2023

1. Itroduction

In 2020, Abbas [1] introduced the notion of h-open sets in a topological space (X, τ) as a generalization of open sets (see also [5]). Acikgoz et al. [3] showed that the family of all h-open sets in (X, τ) is a topology for X (see also [4]).They introduced h-local functions in an ideal topological space (X, τ, I) and obtained their fundamental properties. Quite recently, Acikgoz and Noiri [2] introduced and investigated the notions of hI-open sets, hI-continuous functions and hI-irresolute functions.

In this paper, we introduce $h^\star$-open sets in (X, τ, I), $h^\star$-continuous functions and $h^\star$-irresolute functions. We show that h-open sets and $h^\star$-open sets are independent of each other and they both are stronger than hI-open sets. It is also shown that continuity and $h^\star$-irresoluteness are independent of each other and they both imply $h^\star$–continuity.

2. Preliminaries

Let (X, τ) be a topological space. The notion of ideals has been introduced in [7] and [8] and further investigated in [6].

Definition 2.1. A nonempty collection I of subsets of a set X is called an *ideal on* X if it satisfies the following two conditions:
(1) $A \in I$ and $B \subset A$ implies $B \in I$,
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an *ideal topological space* and is denoted by (X, τ, I). Let (X, τ, I) be an ideal topological space. For any subset A of X, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function* of A with respect to τ and I [6]. Hereafter $A^\star(I, \tau)$ is simply denoted by $A^\star$. It is well known that $\mathrm{Cl}^\star(A) = A \cup A^\star$ defines a Kuratowski closure operator on X and the topology generated by $\mathrm{Cl}^\star$ is denoted by $\tau^\star$.

Definition 2.2. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be
(1) *h-open* [1] if $A \subset Int(A \cup V)$ for every $V \in \tau$ such that $\emptyset \neq V \neq X$,
(2) *hI-open* [2] if $A \subset Int(A \cup Cl^\star(V))$ for every $V \in \tau$ such that $\emptyset \neq V \neq X$.

Lemma 2.3. ([2]) *Every h-open set is hI-open but the converse is not true.*

Definition 2.4. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be
(1) *h-continuous* [1] if for every open set V in Y, $f^{-1}(V)$ is h-open in X,
(2) *hI-continuous* [2] if for every open set V in Y, $f^{-1}(V)$ is hI-open in X.

Lemma 2.5. ([2]) *Every h-continuous function is hI-continuous but the converse is not true.*

3. $h^\star$-open sets

Definition 3.1. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be
(1) *$h^\star$-open* if $A \subset Int(A \cup V^\star)$ for every $V \in \tau$ such that $\emptyset \neq V \neq X$,
(2) *$h^\star$-closed* if $X \setminus A$ is $h^\star$-open.

Let (X, τ, I) be an ideal topological space. I is said to be *codense* if $\tau \cap I = \emptyset$.

Lemma 3.2. ([6]) *Let (X, τ, I) be an ideal topological space. Then the following properties are equivalent:*
(1) I is codense;
(2) $V \subset V^\star$ for every open set V of X.

Theorem 3.3. *Let (X, τ, I) be an ideal topological space. Then the following properties hold:*
(1) If I is codense, then $h^\star$-open sets and hI-open sets are equivalent,
(2) The following diagram holds:

$$\begin{array}{ccc} \textit{open sets} & \Rightarrow & h^\star\textit{-open sets} \\ \Downarrow & & \Downarrow \\ h\textit{-open sets} & \Rightarrow & hI\textit{-open sets} \end{array}$$

(3) h-open sets and $h^\star$-open sets are independent of each other.

Proof. (1) Let V be any open set of X such that $\emptyset \neq V \neq X$. Then, since I is codense, by Lemma 3.1 $V \subset V^\star$ and $\mathrm{Cl}^\star(V) = V \cup V^\star = V^\star$. Hence $h^\star$-open sets and hI-open sets are equivalent.

(2) It is obvious that every open set is h-open and $h^\star$-open. Since $\mathrm{Cl}^\star(V) = V \cup V^\star$, every $h^\star$-open set is hI-open. It is known in [2] that every h-open set is hI-open.

(3) It follows from the following two examples that h-open sets and $h^\star$-open sets are independent of each other.

Example 3.4. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{b\}, \{b, c\}\}, I = \{\emptyset, \{c\}\}$ and $A = \{a, b\}$. Then A is $h^\star$-open and not h-open. For any open set $V \in \tau$ such that $\emptyset \neq V \neq X$, $V^\star = X$ and $A \subset \mathrm{Int}(A \cup V^\star)$ for every open V such that $\emptyset \neq V \neq X$. Therefore, A is $h^\star$-open. There exists an open set $\{b\}$ such that A is not contained in $\mathrm{Int}(A \cup \{b\}) = \{b\}$. Hence A is not h-open.

Example 3.5. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\}, I = \{\emptyset, \{a\}\}$ and $A = \{b, c\}$. Then A is h-open and not $h^\star$-open. For any open set $V \in \tau$ such that $\emptyset \neq V \neq X$, $\mathrm{Int}(A \cup V) = X$. Therefore, A is h-open. There exists an open set $\{a\}$ such that $\{a\}^\star = \emptyset$ and A is not open. Therefore, A is not contained in $\mathrm{Int}(A \cup \{a\}^\star) = \emptyset$. Hence A is not $h^\star$-open.

Let (X, τ, I) be an ideal topological space. The family of all $h^\star$-open sets in (X, τ, I) is denoted by $h^\star O(X, I)$ or simply $h^\star O$.

Theorem 3.6. *Let (X, τ, I) be an ideal topological space. Then $h^\star O$ is a topology for X.*

Proof. (1) It is obvious that $\emptyset, X \in h^\star O$.

(2) Let $V_1, V_2 \in h^\star O$. We show that $V_1 \cap V_2 \in h^\star O$. Let G be any open set of X such that $\emptyset \neq G \neq X$. Since $V_1, V_2 \in h^\star O$, $V_1 \subset \mathrm{Int}(V_1 \cup G^\star)$ and $V_2 \subset \mathrm{Int}(V_2 \cup G^\star)$. Hence $V_1 \cap V_2 \subset \mathrm{Int}(V_1 \cup G^\star) \cap \mathrm{Int}(V_2 \cup G^\star)$

$= \text{Int}\{(V_1 \cup G^\star) \cap (V_2 \cup G^\star)\} \subset \text{Int}\{(V_1 \cap V_2) \cup G^\star)\}$. Therefore, $V_1 \cap V_2 \in h^\star O$.

(3) Let $V_\alpha \in h^\star O$ for each $\alpha \in \Delta$ and G be any open set of X such that $\emptyset \neq G \neq X$. Then $V_\alpha \subset \text{Int}(V_\alpha \cup G^\star)$ for each $\alpha \in \Delta$. Then we have $V_\alpha \subset \text{Int}(V_\alpha \cup G^\star) \subset \text{Int}((\cup_{\alpha\in\Delta} V_\alpha) \cup G^\star)$ for each $\alpha \in \Delta$. Hence $\cup_{\alpha\in\Delta} V_\alpha \subset \text{Int}((\cup_{\alpha\in\Delta} V_\alpha) \cup G^\star)$. This shows that $\cup_{\alpha\in\Delta} V_\alpha \in h^\star O$.

Definition 3.7. Let (X, τ, I) be an ideal topological space and A a subset of X. The set $\cup\{U : U \subset A, U \in h^\star O(X, I)\}$ is called the $h^\star$-*interior* of A and is denoted by $\text{Int}_{h^\star}(A)$.

Theorem 3.8. *Let (X, τ, I) be an ideal topological space. Let A and B be subsets of X. Then the following properties hold:*

(1) If $A \subset B$, then $\text{Int}_{h^\star}(A) \subset \text{Int}_{h^\star}(B)$,
(2) $\text{Int}_{h^\star}(A) \subset A$ and $\text{Int}_{h^\star}(A)$ is $h^\star$-open,
(3) $\text{Int}_{h^\star}(\text{Int}_{h^\star}(A)) = \text{Int}_{h^\star}(A)$,
(4) A is $h^\star$-open if and only if $A = \text{Int}_{h^\star}(A)$,
(5) $\text{Int}_{h^\star}(A) \cap \text{Int}_{h^\star}(B) = \text{Int}_{h^\star}(A \cap B)$,
(6) $\text{Int}_{h^\star}(A) \cup \text{Int}_{h^\star}(B) \subset \text{Int}_{h^\star}(A \cup B)$.

Proof. The proof is obvious.

Definition 3.9. Let (X, τ, I) be an ideal topological space and A a subset of X. The set $\cap\{F : A \subset F, F$ is $h^\star$-*closed* $\}$ is called the $h^\star$-*closure* of A and is denoted by $\text{Cl}_{h^\star}(A)$.

Theorem 3.10. *Let (X, τ, I) be an ideal topological space. Let A and B be subsets of X. Then the following properties hold:*

(1) If $A \subset B$, then $\text{Cl}_{h^\star}(A) \subset \text{Cl}_{h^\star}(B)$,
(2) $A \subset \text{Cl}_{h^\star}(A)$ and $\text{Cl}_{h^\star}(A)$ is $h^\star$-closed,
(3) $\text{Cl}_{h^\star}(\text{Cl}_{h^\star}(A)) = \text{Cl}_{h^\star}(A)$,
(4) A is $h^\star$-closed if and only if $A = \text{Cl}_{h^\star}(A)$,
(5) $\text{Cl}_{h^\star}(A \cap B) \subset \text{Cl}_{h^\star}(A) \cap \text{Cl}_{h^\star}(B)$,
(6) $\text{Cl}_{h^\star}(A \cup B) = \text{Cl}_{h^\star}(A) \cup \text{Cl}_{h^\star}(B)$.

Proof. The proof is obvious.

Theorem 3.11. *Let (X, τ, I) be an ideal topological space and A be a subset of X. Then the following properties hold:*

(1) $X \setminus \text{Cl}_{h^\star}(A) = \text{Int}_{h^\star}(X \setminus A)$,
(2) $X \setminus \text{Int}_{h^\star}(A) = \text{Cl}_{h^\star}(X \setminus A)$.

Proof. The proof is obvious.

4. $h^\star$-continuous functions

Definition 4.1. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be $h^\star$-*continuous* if for every open set V in Y $f^{-1}(V)$ is $h^\star$-open in X.

Remark 4.2. For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following implications hold:

$$\begin{array}{ccc} \text{continuity} & \Rightarrow & h^\star\text{-continuity} \\ \Downarrow & & \Downarrow \\ h\text{-continuity} & \Rightarrow & hI\text{-continuity} \end{array}$$

In the above diagram, $h^\star$-continuity and h-continuity are independent of each other as shown by the following examples.

Example 4.3. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\}, I = \{\emptyset, \{a\}\}, Y = \{a, b, c\}$ and $\sigma = \{\emptyset, Y, \{b, c\}\}$. Then the identity function $f : (X, \tau, I) \to (Y, \sigma)$ is h-continuous and not $h^\star$-continuous. Because, by Example 3.2, $\{b, c\}$ is h-open and not $h^\star$-open in X.

Example 4.4. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{b\}, \{b, c\}\}, I = \{\emptyset, \{a\}\}, Y = X$ and $\sigma = \{\emptyset, Y, \{a, b\}\}$. Then the identity function $f : (X, \tau, I) \to (Y, \sigma)$ is $h^\star$-continuous and not h-continuous. Because, $\{a, b\}$ is $h^\star$-open and not h-open in X.

Lemma 4.5. *Let (X, τ, I) be an ideal topological space and A be a subset of X. Then the following properties are equivalent:*
(1) A is $h^\star$-closed;
(2) $\mathrm{Cl}(A \cap (X \setminus V^\star)) \subset A$ for every open set V of X such that $\emptyset \neq V \neq X$.

Proof. (1) $\Rightarrow$ (2): Let A be $h^\star$-closed. Then $X \setminus A$ is $h^\star$-open. By Definition 3.1, $(X \setminus A) \subset \mathrm{Int}\{(X \setminus A) \cup V^\star\}$ for every open set V of X such that $\emptyset \neq V \neq X$. Therefore, $A \supset X \setminus \mathrm{Int}\{(X \setminus A) \cup V^\star\} = \mathrm{Cl}[X \setminus \{(X \setminus A) \cup V^\star\}] = \mathrm{Cl}[A \cap (X \setminus V^\star)]$. Therefore, we obtain $\mathrm{Cl}(A \cap (X \setminus V^\star)) \subset A$.

(2) $\Rightarrow$ (1): Suppose that $\mathrm{Cl}(A \cap (X \setminus V^\star)) \subset A$ for every open set V of X such that $\emptyset \neq V \neq X$. Then $X \setminus A \subset X \setminus \mathrm{Cl}(A \cap (X \setminus V^\star)) = \mathrm{Int}[X \setminus \{A \cap (X \setminus V^\star)\}] = \mathrm{Int}[(X \setminus A) \cup V^\star]$. Threfore, $X \setminus A$ is $h^\star$-open and hence A is $h^\star$-closed.

Theorem 4.6. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*
(1) f is $h^\star$-continuous;
(2) For each $x \in X$ and each $V \in \sigma$ such that $f(x) \in V$, there exists an $h^\star$-open set U containing x such that $f(U) \subset V$;
(3) For each closed set F in Y, $f^{-1}(F)$ is $h^\star$-closed;
(4) For each closed set F in Y, $\mathrm{Cl}(f^{-1}(F) \cap (X \setminus V^\star)) \subset f^{-1}(F)$ for every open set V of X such that $\emptyset \neq V \neq X$;
(5) For each subset B of Y, $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(B)) \cap (X \setminus V^\star)) \subset f^{-1}(\mathrm{Cl}(B))$ for every open set V of X such that $\emptyset \neq V \neq X$;
(6) For each subset A of X, $f(\mathrm{Cl}(A \cap (X \setminus V^\star))) \subset \mathrm{Cl}(f(A))$ for every open set V of X such that $\emptyset \neq V \neq X$;
(7) For each B of Y, $\mathrm{Cl}_{h^\star}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$;
(8) For each B of Y, $f^{-1}(\mathrm{Int}(B)) \subset \mathrm{Int}_{h^\star}(f^{-1}(B))$.

Proof. (1) $\Rightarrow$ (2): Let x be any point of X and V any open set of Y containing $f(x)$. Set $U = f^{-1}(V)$, then U is an $h^\star$-open set containing x such that $f(U) \subset V$.
(2) $\Rightarrow$ (1): Let V be any open set of Y. For any $x \in f^{-1}(V)$, $f(x) \in V$. By (2), there exists an $h^\star$-open set U_x containing x such that $f(U_x) \subset V$. Since $x \in U_x \subset f^{-1}(V)$, $f^{-1}(V) = \cup\{U_x : x \in f^{-1}(V)\}$ and $f^{-1}(V)$ is $h^\star$-open in X.

(1) $\Rightarrow$ (3): Let F be any closed set of Y. Then $Y \setminus F$ is open in Y and $X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$ is $h^\star$-open in X. Hence $f^{-1}(F)$ is $h^\star$-closed in X.

(3) $\Rightarrow$ (4): Let F be any closed set in Y. Then $f^{-1}(F)$ is $h^\star$-closed in X. By Lemma 4.1, $\mathrm{Cl}(f^{-1}(F) \cap (X \setminus V^\star)) \subset f^{-1}(F)$ for every open set V of X such that $\emptyset \neq V \neq X$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then $\mathrm{Cl}(B)$ is closed in Y and by (4) $\mathrm{Cl}[f^{-1}(\mathrm{Cl}(B)) \cap (X \setminus V^\star)) \subset f^{-1}(\mathrm{Cl}(B))$ for every open set V of X such that $\emptyset \neq V \neq X$.

(5) $\Rightarrow$ (6): Let A be any subset of X. Let $B = f(A)$ in (5). Then $\mathrm{Cl}[A \cap (X \setminus V^\star)] \subset \mathrm{Cl}[f^{-1}(\mathrm{Cl}(f(A))) \cap (X \setminus V^\star)] \subset f^{-1}(\mathrm{Cl}(f(A)))$. Hence $f(\mathrm{Cl}(A \cap (X \setminus V^\star)) \subset \mathrm{Cl}(f(A))$ for every $V \in \tau$ such that $\emptyset \neq V \neq X$.

(6) $\Rightarrow$ (1): Let V be any open set of Y. The $Y \setminus V$ is closed in Y. By (6), for every $V \in \tau$ such that $\emptyset \neq V \neq X$, $f(\mathrm{Cl}[f^{-1}(Y \setminus V) \cap (X \setminus V^\star)] \subset \mathrm{Cl}(f(f^{-1}(Y \setminus V))) \subset \mathrm{Cl}(Y \setminus V) = Y \setminus V$ and hence $\mathrm{Cl}[f^{-1}(Y \setminus V) \cap (X \setminus V^\star)] \subset f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Therefore, we have $f^{-1}(V) \subset X \setminus \mathrm{Cl}[f^{-1}(Y \setminus V) \cap (X \setminus V^\star)] = \mathrm{Int}[X \setminus \{f^{-1}(Y \setminus V) \cap (X \setminus V^\star)\}] = \mathrm{Int}(f^{-1}(V) \cup V^\star)$. Therefore, $f^{-1}(V)$ is $h^\star$-open.

(3) $\Rightarrow$ (7): Let B be any subset of Y. Then $\mathrm{Cl}(B)$ is closed in Y and by (3) $f^{-1}(\mathrm{Cl}(B))$ is $h^\star$-closed. Since $f^{-1}(B) \subset f^{-1}(\mathrm{Cl}(B))$, we obtain $\mathrm{Cl}_{h^\star}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$.

(7) $\Rightarrow$ (8): Let B be any subset of Y. Then we have $f^{-1}(\mathrm{Int}(B)) = f^{-1}(Y \setminus \mathrm{Cl}(Y \setminus B)) = X \setminus f^{-1}(\mathrm{Cl}(Y \setminus B)) \subset X \setminus \mathrm{Cl}_{h^\star}(f^{-1}(Y \setminus B)) = X \setminus \mathrm{Cl}_{h^\star}(X \setminus f^{-1}(B)) = \mathrm{Int}_{h^\star}(f^{-1}(B))$.

(8) $\Rightarrow$ (1): Let V be any open set of Y. By (8), $f^{-1}(V) \subset \mathrm{Int}_{h^\star}(f^{-1}(V)) \subset f^{-1}(V)$ and $\mathrm{Int}_{h^\star}(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is $h^\star$-open.

Definition 4.7. A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be $h^\star$-*irresolute* if for every $h^\star$-open set V in Y $f^{-1}(V)$ is $h^\star$-open in X.

Remark 4.8. For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following implications hold:

$$\begin{array}{ccc} \text{continuity} & \Rightarrow & h^\star\text{-continuity} \\ & & \Uparrow \\ & & h^\star\text{-irresoluteness} \end{array}$$

Remark 4.9. In the above diagram, continuity and $h^\star$irresoluteness are independent of each other as shown by the following two examples.

Example 4.10. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{b\}, \{b, c\}\}, I = \{\emptyset, \{a\}\}$, $Y = \{a, b, c\}$ and $\sigma = \{\emptyset, Y, \{a, b\}\}, J = \{\emptyset, \{c\}\}$. Then the identity function $f : (X, \tau, I) \to (Y, \sigma, J)$ is $h^\star$-iresolute and not continuous.

Proof. 1) Since $\{b\}^\star = \{b, c\}^\star = X$, for every subset A of X, we have $A \subset \mathrm{Int}(A \cup V^\star)$ for every open set V such that $\emptyset \neq V \neq X$. Therefore, every subset of X is $h^\star$-open in X. On the other hand, since $\{a, b\}^\star = Y$, for every subset A of Y, we have $A \subset \mathrm{Int}(A \cup V^\star)$ for every open set V such that $\emptyset \neq V \neq Y$. Therefore, every subset of Y is $h^\star$-open in Y. Threfore, the identity function f is $h^\star$-irresolute.

2) There exists an open set $\{a, b\}$ such that $f^{-1}(\{a, b\}) = \{a, b\}$ is not open in X. Therefore, f is not continuous.

Example 4.11. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\}, I = \{\emptyset, \{a\}\}$, $Y = \{a, b, c\}$ and $\sigma = \{\emptyset, Y, \{a, b\}\}, J = \{\emptyset, \{c\}\}$. Then the identity function $f : (X, \tau, I) \to (Y, \sigma, J)$ is continuous and not $h^\star$-iresolute.

Proof. 1) It is obvious that f is continuous.

2) By Example 4.3, $A = \{b, c\}$ is $h^\star$-open in Y. By Example 3.2, $f^{-1}(A) = A = \{b, c\}$ is not $h^\star$-open in X. Hence f is not $h^\star$-iresolute.

Remark 4.12. In the diagram of Remark 4.2, the converse implications are not always true as shown by the following two examples.

Example 4.13. 1) By Example 4.2, every $h^\star$-continuous function is not always h-continuous and hence every $h^\star$-continuous function is not always continuous.

2) Suppose that $h^\star$-continuity implies $h^\star$-irresoluteness. Then continuity implies $h^\star$-irresoluteness. This is contrary to Example 4.4.

Theorem 4.14. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is $h^\star$-irresolute if and only if $f : (X, h^\star O(X, I)) \to (Y, h^\star O(Y, J))$ is continuous.*

Proof. By Theorem 3.2, $h^\star O(X, I)$ and $h^\star O(Y, J)$ are topologies and the proof is obvious.

Theorem 4.15. *For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following properties are equivalent:*

(1) f is $h^\star$-irresolute;

(2) For each $x \in X$ and each $h^\star$-open set V in Y such that $f(x) \in V$, there exists an $h^\star$-open set U containing x such that $f(U) \subset V$;

(3) For each $h^\star$-closed set F in Y, $f^{-1}(F)$ is $h^\star$-closed in X;

(4) For each B of Y, $\mathrm{Cl}_{h^\star}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_{h^\star}(B))$;

(5) For each B of Y, $f^{-1}(\mathrm{Int}_{h^\star}(B)) \subset \mathrm{Int}_{h^\star}(f^{-1}(B))$.

Proof. The proof is similar to Theorem 4.1.

Definition 4.16. A function $f : (X, \tau) \to (Y, \sigma, J)$ is said to be
(1) $h^\star$-*closed* if for every closed set F in X, $f(F)$ is $h^\star$-closed in Y,
(2) $h^\star$-*open* if for every open set U in X, $f(U)$ is $h^\star$-open in Y.

Theorem 4.17. *For a surjective function $f : (X, \tau) \to (Y, \sigma, J)$, the following properties hold:*
(1) f is $h^\star$-closed if and only if for each subset $S \subset Y$ and each open set U in X containing $f^{-1}(S)$, there exists an $h^\star$-open set V in Y such that $S \subset V$ and $f^{-1}(V) \subset U$.
(2) f is $h^\star$-open if and only if for each subset $S \subset Y$ and each closed set U in X containing $f^{-1}(S)$, there exists an $h^\star$-closed set V in Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof. (1) Let S be any subset of Y and U any open set in X containing $f^{-1}(S)$. Then $X \setminus U \subset X \setminus f^{-1}(S) = f^{-1}(Y \setminus S)$. Hence $f(X \setminus U) \subset Y \setminus S$ and $f(X \setminus U)$ is $h^\star$-closed. Set $V = Y \setminus f(X \setminus U)$, then V is $h^\star$-open in Y, $S \subset V$ and $f^{-1}(V) \subset U$.
Conversely, for any closed set F in X, set $U = Y \setminus f(F)$. Then $f^{-1}(U) \subset X \setminus F$ and $X \setminus F$ is open in X. Therefore, there exists an $h^\star$-open set V in Y such that $U \subset V$ and $f^{-1}(V) \subset X \setminus F$. Since $U = Y \setminus f(F)$, $Y \setminus f(F) \subset V$ and $f^{-1}(Y \setminus f(F)) \subset f^{-1}(V) \subset X \setminus F$. Hence $F \subset X \setminus f^{-1}(V) \subset f^{-1}(f(F))$. Since f is surjective, $f(F) \subset Y \setminus V \subset f(F)$ and hence $f(F) = Y \setminus V$ is $h^\star$-closed.
(2) The proof of (2) is similar with (1).

Remark 4.18. The assumption "surjective" in Theorem 4.4 is necessary for the proof of sufficiency.

References

[1] F. Abbas, *On H-open sets and H-continuous functions*, J. Appl. Comput. Math. **9(1)** (2020), 1–5.
[2] A. Acikgoz and T. Noiri, *Some extensions of open sets by idealizations*, submitted.
[3] A. Acikgoz, T. Noiri and B. Golpinar, *On h-local functions in ideal topological spaces*, Mathematica (Cluj) **65(88)(1)** (2023), 3–11.
[4] F.I. Dagci and H. Cakalli, *A new topology via a topology*, AIP Conference Proceedings, AIP Publishing 2483 (2022), 020003.
[5] H. Cakalli and F.I. Dagci, *On H-open sets and H-continuous functions*, J. Appl. Comput. Math. **10(5)** (2021).
[6] D. Janković and T.R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly **97** (1990), 295–310.
[7] K. Kuratowski, *Topology*, Vol. 1, Academic Press, New York, 1966.
[8] R. Vaidyanathswamy, *The localization theory in set topology*, Proc. Indian Acad. Sci. **20** (1945), 51–61.

Department of Mathematics, Balikesir University, 10145 Balikesir, Turkey
Email address: ahuacikgoz@gmail.com

2949-1, Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken, 869-5142, Japan
Email address: t.noiri@nifty.com