EFFECTS OF FUZZY SETTING IN KOROVKIN THEORY VIA P_p-STATISTICAL CONVERGENCE

TUĞBA YURDAKADIM AND EMRE TAŞ

ABSTRACT. In this paper our aim is to approximate a function with the use of fuzzy positive linear operators when the fuzzy limit fails by defining the fuzzy analog of P_p-statistical convergence. It is effective to use this type of convergence since a sequence can still be P_p-statistical convergent while it is neither convergent nor statistically convergent. By considering fuzzy positive linear operators, we obtain Korovkin type approximation results for these operators in the sense of P_p-statistical convergence. The rate of approximation by fuzzy modulus of continuity is also presented.

Mathematics Subject Classification (2010): 03E72, 40C15, 40G15, 41A36
Key words: Fuzzy Korovkin theory, fuzzy rate, power series method, statistical convergence, fuzzy logic.

Article history:
Received: February 9, 2022
Received in revised form: May 29, 2022
Accepted: May 30, 2022

1. INTRODUCTION AND PRELIMINARIES

The class of smart boys, the class of clever students or the class of all apples which are red enough do not construct sets in the usual mathematical sense of these terms. However, the fact lies under such uncertainly defined sets play significant role in human thinking, pattern recognition, machine learning. This motivates Zadeh [30] to define fuzzy sets by assigning to each element a grade of membership ranging from 0 to 1. It is effective to use membership function to overcome the uncertainty. Later, many researchers have extended the well known concepts of classical set theory to fuzzy setting. There are also many studies on fuzzy topology since it is applicable to quantum particle physics [21], [22]. Recently the generalizations of fuzzy topology such as intuitionistic fuzzy topology, Pythagorean fuzzy topology have been studied in [10], [24], [29]. Furthermore fuzzy logic has also been used in different areas of mathematics, for example, while studying metric and topological spaces [28], matrix and linear systems [8], [25], approximation theory. Gal has presented some results dealing with approximation theory in fuzzy setting [18]. Korovkin type approximation results in fuzzy setting by using different types of convergences instead of ordinary convergence have been presented in [1], [2], [3], [9]. Statistical approximation of fuzzy trigonometric functions and fuzzy differentiable functions have been studied in [4], [5], [12], [13]. The corresponding statistical rates in the fuzzy approximation have been obtained in [14]. In ordinary convergence, all of the terms of the sequence except finite number have to belong to an arbitrarily small neighborhood of the limit. This is a critical weakness of ordinary convergence and by flexing this condition only for a majority of elements, statistical convergence has been defined. The aim of obtaining stronger results than the classical ones, different types of convergences have been defined and used in approximation theory.
In this study, by considering fuzzy positive linear operators we present some Korovkin type approximation

results with the use of P_p-statistical convergence. We also obtain the rate of this approximation by fuzzy modulus of continuity. Furthermore we construct examples to show the strength of our results.

Now let us recall the basic definitions and notations.
If the limit

$$\delta(K) := \lim_{k\to\infty} \frac{1}{k+1} |\{n \leq k : n \in K\}|$$

exists then it is said to be the density of the subset $K \subseteq \mathbb{N}_0$. Here by $|.|$, we denote the number of the elements of enclosed set and $\mathbb{N}_0$ is the set of all nonnegative integers. If for every $\varepsilon > 0$, $\delta(K_\varepsilon) = 0$ where $K_\varepsilon = \{n \in \mathbb{N}_0 : |x_n - l| \geq \varepsilon\}$, then it is said that $x = (x_n)$ converges statistically to l [16], [17], [26].

Let (p_n) be a real sequence such that $p_0 > 0$, p_1, p_2, ...≥ 0, and $p(t) := \sum_{n=0}^{\infty} p_n t^n$ has radius of convergence R with $0 < R \leq \infty$. If the limit

$$\lim_{t\to R^-} \frac{1}{p(t)} \sum_{n=0}^{\infty} x_n p_n t^n = l$$

exists then it is said that $x = (x_n)$ is convergent to l in the sense of power series method [7], [20]. The next example shows that ordinary convergence is not as effective as power series method, i.e., power series method is more useful. Let $x = (1, -1, 1, -1, ...)$, $R = \infty$, $p(t) = e^t$ and for $n \geq 0$, $p_n = \frac{1}{n!}$. Then we immediately see that

$$\lim_{t\to\infty} \frac{1}{e^t} \sum_{n=0}^{\infty} \frac{x_n t^n}{n!} = \lim_{t\to\infty} \frac{1}{e^t} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(n)!} = \lim_{t\to\infty} \frac{1}{e^t} e^{-t} = 0.$$

Hence while the sequence $x = (x_n)$ converges to 0 in the sense of power series method, it does not converge in the ordinary sense.
If $\lim x = l$ implies $P_p - \lim x = l$, then it is said that P_p is regular [7]. The regularity of power series method is equivalent to

$$\lim_{t\to R^-} \frac{p_n t^n}{p(t)} = 0$$

holds for each $n \in \mathbb{N}_0$ [7].

Let P_p be regular and $K \subset \mathbb{N}_0$. If the limit

$$\delta_{P_p}(K) := \lim_{t\to R^-} \frac{1}{p(t)} \sum_{n\in K} p_n t^n$$

exists then it is said to be the P_p-density of K.

The sequence $x = (x_n)$ of real numbers P_p-statistically converges to l if for every $\varepsilon > 0$, $\delta_{P_p}(K_\varepsilon) = 0$ that is for every $\varepsilon > 0$

$$\lim_{t\to R^-} \frac{1}{p(t)} \sum_{n\in K_\varepsilon} p_n t^n = 0.$$

An example of a sequence such that statistical convergent but not P_p-statistical convergent and an example of a sequence such that P_p-statistical convergent but not statistically convergent have been presented in [27].
If the followings are satisfied for a function $\nu : \mathbb{R} \longrightarrow [0, 1]$

- ν is normal, i.e., there exists $x_0 \in \mathbb{R}$ such that $\nu(x_0) = 1$,

- ν is convex, i.e., $\nu(\lambda x + (1-\lambda)y) \geq \min\{\nu(x), \nu(y)\}, \ for\ all\ x, y \in \mathbb{R},\ \gamma \in [0, 1]$

- upper semi-continuous on $\mathbb{R}$ and
- the closure of the set $supp(\nu)$ is compact, where

$$supp(\nu) := \{x \in \mathbb{R} : \nu(x) > 0\}$$

then ν is said to be a fuzzy number and $\mathbb{R}_{\mathbb{F}}$ denotes the set of such elements.
Let

$$[\nu]^0 := \overline{\{x \in \mathbb{R} : \nu(x) > 0\}} \; and \; [\nu]^r := \{x \in \mathbb{R} : \nu(x) \geq r\}, (0 < r \leq 1).$$

Recall from [19] that, for each $r \in [0,1]$, the set $[\nu]^r$ is an interval which is closed and bounded in $\mathbb{R}$. For any $q, s \in \mathbb{R}_{\mathbb{F}}$ and $\gamma \in \mathbb{R}$, the operations sum $q \oplus s$ and product $\gamma \odot q$ can be defined uniquely as follows:

$$[q \oplus s]^r = [q]^r + [s]^r \; and \; [\gamma \odot q]^r = \gamma [q]^r, 0 \leq r \leq 1.$$

The interval $[q]^r$ can be denoted by $[q_-^{(r)}, q_+^{(r)}]$ where $q_-^{(r)} \leq q_+^{(r)}$ and $q_-^{(r)}, q_+^{(r)} \in \mathbb{R}$ for $r \in [0,1]$. Then define the following for $q, s \in \mathbb{R}_{\mathbb{F}}$

$$q \preceq s \leftrightarrow q_-^{(r)} \leq s_-^{(r)} \; and \; q_+^{(r)} \leq s_+^{(r)}, \; for \; all \; 0 \leq r \leq 1.$$

On the other hand consider the following metric

$$d : \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \longrightarrow \mathbb{R}_+$$

by

$$d(q,s) = \sup_{r \in [0,1]} \max\{|q_-^{(r)} - s_-^{(r)}|, |q_+^{(r)} - s_+^{(r)}|\}.$$

Note that $(\mathbb{R}_{\mathbb{F}}, d)$ is complete. Then for the fuzzy number valued functions f, g defined on $[a,b]$, the distance is introuced by

$$d^*(f,g) = \sup_{x \in [a,b]} \sup_{r \in [0,1]} \max\{|f_-^{(r)} - g_-^{(r)}|, |f_+^{(r)} - g_+^{(r)}|\}.$$

By using this metric, the statistical convergence has been introduced in fuzzy setting in [23] as follows: Let $(\nu_n)_{n \in \mathbb{N}_0}$ be a sequence of fuzzy numbers. If for every $\varepsilon > 0$,

$$\lim_k \frac{|n \leq k : d(\nu_n, \nu) \geq \varepsilon|}{k+1} = 0$$

holds then it is said that $(\nu_n)_{n \in \mathbb{N}_0}$ converges statistically to ν and we denote it by

$$st - \lim_n d(\nu_n, \nu) = 0.$$

Then in [3], A-statistical convergence has also been defined in fuzzy setting as follows: we say that $(\nu_n)_{n \in \mathbb{N}}$ converges A-statistically to $\nu \in \mathbb{R}_{\mathbb{F}}$ and we denote it by

$$st_A - \lim_n d(\nu_n, \nu) = 0,$$

if for every $\varepsilon > 0$

$$\lim_j \sum_{n : d(\nu_n, \nu) \geq \varepsilon} a_{jn} = 0$$

holds. If $A = C_1$, the Cesáro matrix of order one, then we get statistical convergence recalled above. Again in the case A is the identity matrix, then we get fuzzy convergence.
The main tool of the paper is P_p-statistical convergence and now we are ready to define it in fuzzy setting. If

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{n \in K_\varepsilon} p_n t^n = 0$$

holds for every $\varepsilon > 0$ then it is denoted by $st_{P_p} - \lim d(\nu_n, \nu) = 0$ where $K_\varepsilon = \{n : d(\nu_n, \nu) \geq \varepsilon\}$.

2. Fuzzy Korovkin Theory in P_p-Statistical Sense

This section is devoted to our main results dealing with Korovkin type approximation and the P_p-statistical rate of approximation. We also provide examples to illustrate that it is still possible to approximate a function by fuzzy positive linear operators when the fuzzy limit fails. Therefore it is benefical to recall some of the well known concepts in fuzzy setting.
Let f be a function defined on $[a, b]$ with fuzzy number values. Then the fuzzy continuity of f at $x_0 \in [a, b]$ is defined as follows: if $x_n \to x_0$, then $d(f(x_n), f(x_0)) \to 0$ as $n \to \infty$. If f is continuous at every point $x \in [a, b]$, then it is said that f is fuzzy continuous on $[a, b]$. $C_{\mathbb{F}}[a, b]$ is the set of all fuzzy continuous functions on $[a, b]$. It is important to recall that $C_{\mathbb{F}}[a, b]$ is not a vector space but a cone. Now let $T : C_{\mathbb{F}}[a, b] \longrightarrow C_{\mathbb{F}}[a, b]$ be an operator. If for every $\alpha, \beta \in \mathbb{R}$, $f, g \in C_{\mathbb{F}}[a, b]$ and $x \in [a, b]$,

$$T(\alpha \odot f \oplus \beta \odot g; x) = \alpha \odot T(f; x) \oplus \beta \odot T(g; x)$$

holds then it is said that T is fuzzy linear. Also T is called fuzzy positive linear operator if it is fuzzy linear and $T(f; x) \preceq T(g; x)$ whenever $f, g \in C_{\mathbb{F}}[a, b]$, and all $x \in [a, b]$ with $f(x) \preceq g(x)$.
The following Korovkin type theorem in fuzzy setting has been given by Anastassiou [2].

Theorem 2.1. *Let T_n be fuzzy positive linear operators for every $n \in \mathbb{N}$ from $C_{\mathbb{F}}[a, b]$ into itself. Suppose that there exists a corresponding positive linear operators $\tilde{T_n}$ from $C[a, b]$ into itself with the property*

$$\{T_n(f; x)\}_{\pm}^{(r)} = \{\tilde{T_n}\}(f_{\pm}^{(r)}; x)$$

for all $x \in [a, b]$, $r \in [0, 1], n \in \mathbb{N}$, $f \in C_{\mathbb{F}}[a, b]$. If

$$\lim_n ||\{\tilde{T_n}\}(x^i) - x^i|| = 0, i = 0, 1, 2,$$

then for all $f \in C_{\mathbb{F}}[a, b]$, we have

$$\lim_n d^*(T_n(f), f) = 0.$$

Anastassiou and Duman have given the A-statistical analog of this theorem in [3]. Now it is time to give our main result.

Theorem 2.2. *Let P_p be regular and T_n be fuzzy positive linear operators for every $n \in \mathbb{N}_0$ from $C_{\mathbb{F}}[a, b]$ into itself. Suppose that there exists a corresponding positive linear operators $\tilde{T_n}$ from $C[a, b]$ into itself with the property*

$$\{T_n(f; x)\}_{\pm}^{(r)} = \{\tilde{T_n}\}(f_{\pm}^{(r)}; x)$$

for all $x \in [a, b]$, $r \in [0, 1], n \in \mathbb{N}_0$, $f \in C_{\mathbb{F}}[a, b]$. If

$$st_{P_p} - \lim_n \|\{\tilde{T_n}\}(x^i) - x^i\| = 0, i = 0, 1, 2,$$

then for all $f \in C_{\mathbb{F}}[a, b]$, we have

$$st_{P_p} - \lim_n d^*(T_n(f), f) = 0.$$

Proof. Let $f \in C_{\mathbb{F}}[a, b]$, $x \in [a, b]$ and $r \in [0, 1]$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}(x)| < \varepsilon$ holds for every $y \in [a, b]$ satisfying $|y - x| < \delta$ since $f_{\pm}^{(r)} \in C[a, b]$. As in classical Korovkin theory, we have that

$$|f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}(x)| \leq \varepsilon + 2H_{\pm}^{(r)} \frac{(y - x)^2}{\delta^2}$$

holds for all $y \in [a, b]$ where $2H_{\pm}^{(r)} := \|2f_{\pm}^{(r)}\|$. $\{\tilde{T_n}\}$ is positive and linear,

$$\begin{aligned} |\{\tilde{T_n}\}(f_{\pm}^{(r)}; x) - f_{\pm}^{(r)}(x)| &\leq \{\tilde{T_n}\}(|f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}|; x) + H_{\pm}^{(r)}|\{\tilde{T_n}\}(1; x) - 1| \\ &\leq \varepsilon + (\varepsilon + H_{\pm}^{(r)}|\{\tilde{T_n}\}(1; x) - 1|) + \frac{2H_{\pm}^{(r)}}{\delta^2}|\{\tilde{T_n}\}((y - x)^2; x)| \end{aligned}$$

holds for each $n \in \mathbb{N}_0$ and it implies

$$|\{\tilde{T_n}\}(f_{\pm}^{(r)};x)-(f_{\pm}^{(r)};x)|\leq \varepsilon+(\varepsilon+H_{\pm}^{(r)}+2h^2\frac{H_{\pm}^{(r)}}{\delta^2})|\{\tilde{T_n}\}(1;x)-e_1| + 4h\frac{H_{\pm}^{(r)}}{\delta^2}|\{\tilde{T_n}\}(t;x)-x|+2\frac{H_{\pm}^{(r)}}{\delta^2})|\{\tilde{T_n}\}(t^2;x)-x^2|$$

where $h:=\max\{|a|,|b|\}$. Pick

$$H_{\pm}^{(r)}(\varepsilon):=\max\{\varepsilon+H_{\pm}^{(r)}+2h^2\frac{H_{\pm}^{(r)}}{\delta^2},4h\frac{H_{\pm}^{(r)}}{\delta^2},2\frac{H_{\pm}^{(r)}}{\delta^2}\}$$

and take supremum over $x\in[a,b]$, then we have that

$$\|\{\tilde{T_n}\}(f_{\pm}^{(r)})-f_{\pm}^{(r)}\|\leq\varepsilon+H_{\pm}^{(r)}(\varepsilon)\{\|\{\tilde{T_n}\}(1)-1\|+\|\{\tilde{T_n}\}(x)-x\|+\|\{\tilde{T_n}\}(x^2)-x^2\|\}.$$

Then, by the property in hypothesis, we obtain that

$$\begin{aligned}d^*(T_n(f),f) &= \sup_{x\in[a,b]} d(T_n(f;x)-f(x))\\ &= \sup_{x\in[a,b]}\sup_{r\in[0,1]}\max\{|\{\tilde{T_n}\}(f_-^{(r)};x)-f_-^{(r)}(x)|,|\{\tilde{T_n}\}(f_+^{(r)};x)-f_+^{(r)}(x)\}\\ &= \sup_{r\in[0,1]}\max\{\|\{\tilde{T_n}\}(f_-^{(r)})-(f_-^{(r)})\|,\|\{\tilde{T_n}\}(f_+^{(r)})-(f_+^{(r)})\|\}.\end{aligned}$$

Considering the above inequalities, we obtain that

$$d^*(T_n(f),f)\leq\varepsilon+H(\varepsilon)\{\|\{\tilde{T_n}\}(1)-1\|+\|\{\tilde{T_n}\}(x)-x\|+\|\{\tilde{T_n}\}(x^2)-x^2\|\}$$

where $H(\varepsilon):=\sup_{r\in[0,1]}\max\{H_-^{(r)}(\varepsilon),H_+^{(r)}(\varepsilon)\}$. Now for a given $\varepsilon^{'}$, choose $\varepsilon>0$ such that $0<\varepsilon<\varepsilon^{'}$ and also define

$$K:=\{n\in\mathbb{N}_0: d^*(T_n(f),f)\geq\varepsilon^{'}\},$$

$$K_0:=\{n\in\mathbb{N}_0:\|\{\tilde{T_n}\}(1)-1\|\geq\frac{\varepsilon^{'}-\varepsilon}{3H(\varepsilon)}\},$$

$$K_1:=\{n\in\mathbb{N}_0:\|\{\tilde{T_n}\}(x)-x\|\geq\frac{\varepsilon^{'}-\varepsilon}{3H(\varepsilon)}\},$$

$$K_2:=\{n\in\mathbb{N}_0:\|\{\tilde{T_n}\}(x^2)-x^2\|\geq\frac{\varepsilon^{'}-\varepsilon}{3H(\varepsilon)}\}.$$

Using the above inequalities, we have $K\subseteq K_0\cup K_1\cup K_2$ which implies that

$$\frac{1}{p(t)}\sum_{n\in K}p_nt^n\leq\frac{1}{p(t)}\{\sum_{n\in K_0}p_nt^n+\sum_{n\in K_1}p_nt^n+\sum_{n\in K_2}p_nt^n\}.$$

By taking limit as $0<t\to R^-$ on the both sides and using the hypothesis, we immediately obtain that

$$\lim_{0<t\to R^-}\frac{1}{p(t)}\sum_{n\in K}p_nt^n=0.$$

Hence the proof is completed. □

Example 2.3. *Let the sequences (p_n) and (a_n) defined as follows:*

$$p_n=\begin{cases}1 & , \quad n=2k\\ 0 & , \quad n=2k+1\end{cases}, \qquad a_n=\begin{cases}0 & , \quad n=2k+1\\ 1 & , \quad n=2k\end{cases}.$$

One can immediately obtain that the method P_p is regular and

$$\delta_{P_p}(K_\varepsilon)=0$$

where $K_\varepsilon = \{n \in \mathbb{N}_0 : |a_n - 1| \geq \varepsilon\}$ *holds for every* $\varepsilon > 0$. *That is* $st_{P_p} - \lim a_n = 1$. *Notice that* (a_n) *is neither convergent in the ordinary sense nor statistically convergent. Now construct the fuzzy Bernstein-type operators as follows*

$$T_n^{\mathbb{F}}(f;x) = \begin{cases} a_n \odot \oplus_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \odot f(\frac{k}{n}) & , \quad n \in \mathbb{N} \\ f(x) & , \quad n = 0 \end{cases}$$

where $f \in C_{\mathbb{F}}[0,1],\ x \in [0,1]$.
In this case, one can also write

$$\{T_n^{\mathbb{F}}(f;x)\}_{\pm}^{r} = \{\tilde{T_n}\}(f_{\pm}^{(r)};x) = a_n \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f_{\pm}^{(r)}(\frac{k}{n})$$

where $f_{\pm}^{(r)} \in C[0,1]$. *Notice that*

$$\{\tilde{T_n}\}(1;x) = a_n,$$
$$\{\tilde{T_n}\}(t;x) = x a_n,$$
$$\{\tilde{T_n}\}(t^2;x) = [x^2 + \frac{x(1-x)}{n}]a_n.$$

Then

$$st_{P_p} - \lim_n \|\{\tilde{T_n}\}(x^i) - x^i\| = 0$$

holds for $i = 0, 1, 2$ *then*

$$st_{P_p} - \lim_n d^*(T_n^{\mathbb{F}}(f), f) = 0$$

holds for all $f \in C_{\mathbb{F}}[a,b]$ *follows from our main result. Notice that since the sequence* (a_n) *is not convergent,* $\{T_n^{\mathbb{F}}(f)\}_{n \in \mathbb{N}_0}$ *is not fuzzy convergent to* f.

Example 2.4. *Let* (p_n) *and* (a_n) *be defined as follows:*

$$p_n = \begin{cases} 1 & , \quad n = 2k \\ 0 & , \quad n = 2k+1 \end{cases} , \qquad a_n = \begin{cases} 0 & , \quad n = 2k \\ 1 & , \quad n = 2k+1 \end{cases} .$$

It is easy to see that P_p *is regular and*

$$K_\varepsilon = \{n \in \mathbb{N}_0 : |a_n - 0| \geq \varepsilon\} \subseteq \{n = 2k+1 : k \in \mathbb{N}_0\}$$

holds for every $\varepsilon > 0$. *Then we have*

$$\delta_{P_p}(K_\varepsilon) = \lim_{0<t \to R^-} \frac{1}{p(t)} \sum_{n \in K_\varepsilon} p_n t^n = 0$$

i.e., that (a_n) *is* P_p*-statistically convergent to 0. Construct the following fuzzy Bernstein-type operators:*

$$T_n^{\mathbb{F}}(f;x) = \begin{cases} (1+a_n) \odot \oplus_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \odot f(\frac{k}{n}) & , \quad n \in \mathbb{N} \\ f(x) & , \quad n = 0 \end{cases}$$

where $f \in C_{\mathbb{F}}[0,1],\ x \in [0,1]$.
Notice that since the sequence (a_n) *is not convergent,* $\{T_n^{\mathbb{F}}(f)\}_{n \in \mathbb{N}_0}$ *is not fuzzy convergent to* f *but still one can approximate* f *by* $T_n^{\mathbb{F}}(f)$ *with the use of* P_p*-statistical convergence.*

Now recall the modulus of continuity in fuzzy setting. Let f be a function defined on $[a,b]$ and fuzzy number valued. Then the fuzzy modulus of continuity of f is defined in [18] as follows:

$$w_1^{\mathbb{F}} := \sup_{x \in [a,b]:|x-y| \leq \delta} d(f(x), f(y))$$

for any $0 < \delta \leq b - a$. The rates of this approximation have been presented in [3] by this notion. Statistical rate of convergence has been defined and studied in [11], [15]. By modificating these concepts, P_p- statistical rate of convergence has been introduced in [6].

Definition 2.5. *Let (a_n) be a positive non-increasing sequence of real numbers and let P_p be regular. A sequence $x=(x_n)$ is P_p-statistically converges to the number l with rate $o(a_n)$ if for every $\varepsilon>0$*

$$\lim_{0<t\to R^-}\left[\frac{1}{p(t)}\sum_{n:|x_n-l|\geq\varepsilon a_n} p_n t^n\right]=0$$

and we denote it by $x_n-l=st_{P_p}-o(a_n)$, as $n\to\infty$.

Theorem 2.6. *Let P_p be regular and T_n be fuzzy positive linear operators $n\in\mathbb{N}_0$ from $C_{\mathbb{F}}[a,b]$ into itself. Suppose that there exists a corresponding positive linear operators $\tilde{T_n}$ of from $C[a,b]$ into itself with the property*

$$\{T_n(f;x)\}_{\pm}^{(r)}=\{\tilde{T}\}_n(f_{\pm}^{(r)};x)$$

for all $x\in[a,b]$, $r\in[0,1], n\in\mathbb{N}_0$, $f\in C_{\mathbb{F}}[a,b]$. If $(a_n),(b_n)$ are positive non-increasing sequences and also the operators $\{\tilde{T_n}\}$ satisfy the following conditions:

$$\|\{\tilde{T_n}\}(1)-1\|=st_{P_p}-o(a_n)$$

$$w_1^{\mathbb{F}}(f,\nu_n)=st_{P_p}-o(b_n),$$

then for all $f\in C_{\mathbb{F}}[a,b]$, we have

$$d^*(T_n(f),f)=st_{P_p}-o(c_n).$$

Here $\nu_n:=\sqrt{\|\{\tilde{T_n}\}(\phi)\|}$, $\phi(y)=(y-x)^2$ for each $x\in[a,b]$ and $c_n=\max\{a_n,b_n,a_nb_n\}$, for every $n\in\mathbb{N}_0$.

Proof. By Theorem 3 of [2], one can get, for each $n\in\mathbb{N}_0$ and $f\in C_{\mathbb{F}}[a,b]$, that

$$d^*(T_n(f),f)\leq H\|\{\tilde{T_n}\}(1)-1\|+\|\{\tilde{T_n}\}(1)+1\|w_1^{\mathbb{F}}(f,\nu_n)$$

where $H:=d^*(f,\chi_0)$ and χ_0 denotes the neutral element for $\oplus$. Then we have that

$$d^*(T_n(f),f)\leq H\|\{\tilde{T_n}\}(1)-1\|+\|\{\tilde{T_n}\}(1)+1\|w_1^{\mathbb{F}}(f,\nu_n)+2w_1^{\mathbb{F}}(f,\nu_n).$$

By using the similar idea in Lemma 4 in [11] and taking care of the right hand side of the following equality we obtain the desired result. Therefore the proof is completed. ☐

3. Conclusion

The fact lying under uncertainly defined sets play important role in human thinking, pattern recognition and machine learning and this motivates Zadeh to introduce fuzzy sets by attaching a grade of membership to each element. Since it is effective to overcome uncertainty, fuzzy theory has become an active area of research. Fuzzy logic and fuzzy settings of well-known concepts have been studied. In the present paper, by considering fuzzy positive linear operators we have obtained Korovkin type approximation results via P_p-statistical convergence. We have also studied the rate of this approximation with the use of fuzzy modulus of continuity. It is important to mention that our results are stronger than the results in the existing literature since P_p-statistical convergence is flexing the critical weakness of ordinary convergence. In order to show the strength of our results, we have provided some examples.

References

[1] M. Aiyub, K. Saini and K. Raj, *Korovkin type approximation theorem via lacunary equi-statistical convergence in fuzzy spaces*, J. Math. Computer Sci. **25** (2022), 312-321.
[2] G.A. Anastassiou, *On basic fuzzy Korovkin theory*, Stud. Univ. Babes-Bolyai Math. **50** (2005), 3-10.
[3] G.A. Anastassiou and O. Duman, *Statistical fuzzy approximation by fuzzy positive linear operators*, Comput. and Math. Appl. **55** (2008), 573-580.
[4] G.A. Anastassiou and O. Duman, *High order statistical fuzzy Korovkin theory*, Stoch. Anal. Appl. **27(3)** (2009), 543-554.

[5] G.A. Anastassiou and O. Duman, *Towards Intelligent Modelling: Statistical approximation theory*, Intelligent Systems Reference Library, Springer-Verlag, Berlin **14** (2011).
[6] Ö.G. Atlıhan, T. Yurdakadim and E. Taş, *A new approach to approximation by positive linear operators in weighted spaces*, Accepted for publication in Ukrainian J. Math.
[7] J. Boos, *Classical and Modern Methods in Summability*, Oxford University Press, 2000.
[8] J.J. Buckley, *Solving systems of linear fuzzy equations*, Fuzzy Sets and Syst. **43** (1991), 33-43.
[9] M. Burgin and O. Duman, *Approximations by linear operators in spaces of fuzzy continuous functions*, Positivity **15** (2011), 57-72.
[10] D. Coker, *An introduction to intuitionistic fuzzy topological spaces*, Fuzzy Sets and Syst. **88** (1997), 81-89.
[11] O. Duman, M.K. Khan and C. Orhan, *A-statistical convergence of approximating operators*, Math. Inequal. Appl. **6** (2003), 689-699.
[12] O. Duman and G.A. Anastassiou, *On statistical fuzzy trigonometric Korovkin theory*, J. Comput. Anal. Appl. **10(3)** (2008), 333-344.
[13] O. Duman, *Statistical fuzzy approximation to fuzzy differentiable functions by fuzzy linear operators*, Hacet. J. Math. Stat. **39(4)** (2010), 497-514.
[14] O. Duman, *Fuzzy approximation based on statistical rates*, Publ. Math. Debrecen **76(3-4)** (2010), 453-464.
[15] J.A. Fridy, H.I. Miller and C. Orhan, *Statistical rates of convergence*, Acta Sci. Math. **69** (2003), 147-157.
[16] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241-244.
[17] J.A. Fridy, *On statistical convergence*. Analysis **5** (1985), 301-313.
[18] S. Gal, *Approximation theory in fuzzy setting. Chapter 13 in Handbook of Analytic-Computational Methods in Applied Mathematics*, Chapman and Hall/CRC, Boca Raton (2000), 617-666.
[19] R.J. Goetschel and W. Voxman, *Elementary fuzzy calculus*, Fuzzy Sets and Syst. **18** (1986), 31-43.
[20] W. Kratz and U. Stadtmüller, *Tauberian theorems for J_p-summability*, J. Math. Anal. Appl. **139** (1989), 362-371.
[21] M.S. El Naschie, *On uncertainty of Cantorian geometry and two-slit experiment*, Chaos Solitons and Fractals **9** (1998), 517-529.
[22] M.S. El Naschie, *A review of E-infinity theory and mass spectrum of high energy particle physics*, Chaos Solitons and Fractals **19** (2004), 209-236.
[23] F. Nuray and E. Savaş, *Statistical convergence of sequences of fuzzy numbers*, Math. Slovaca. **45** (1995), 269-273.
[24] M. Olgun, M. Ünver and Ş. Yardımc ı, *Pythagorean fuzzy topological spaces*, Complex and Intelligent Systems **5** (2019), 177-183.
[25] K. Peeva, *Fuzzy linear systems*, Fuzzy Sets and Syst. **49** (1992), 339-355.
[26] T. Salat, *On statistically convergent sequences of real numbers*, Mat. Slovaca. **30** (1980), 139-150.
[27] M. Ünver and C. Orhan, *Statistical convergence with respect to power series methods and applications to approximation theory*, Numer. Funct. Anal. Optim. **40** (2019), 535-547.
[28] A. George and P. Veeramani, *On some results in fuzzy metric space*. Fuzzy Sets and Syst. **4** (1980), 395-399.
[29] R.R. Yager, *Pythagorean fuzzy subsets*, In:2013 joint IFSA world congres and NAFIPS annual meeting (IFSA/NAFIPS), IEEE, Edmenton, AB, Canada, 24-28 June 2013, 57-61.
[30] L.A. Zadeh, *Fuzzy sets* Inform. and Control **8** (1965), 338-353.

Bilecik Şeyh Edebali University, Department of Mathematics, Bilecik 11230, Turkey
Email address: tugbayurdakadim@hotmail.com

Kirşehir Ahi Evran University, Department of Mathematics, Kirşehir 40100, Turkey
Email address: emretas86@hotmail.com